A regularized elliptic-parabolic model for the transport in porous media

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From previous works with C. Le Potier (CEA) and from the PhD of C. Baudry

Outline

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Model of porous media and Richards' equation

- Basic equations
- Retention laws and degeneracy

Elliptic-parabolic non degenerate problem

- Mechanical models
- Ellipticity

Conclusions

Darcy's law and hydraulic head

Bernouilli equation (equilibrium):

$\frac{1}{2}(\vec{u})^2 + \rho g z + p = C$, C constant

Definition of hydraulic head:

$$h = \frac{p}{\rho g} + z + \frac{(\vec{u})^2}{2\rho g} \simeq \frac{p}{\rho g} + z.$$

Constant in a perfect fluid (not in a porous medium). Case of compressible fluid $h = z + \int_{p_0}^{p} \frac{dp}{\rho(p)g}$.

Darcy law (from Poiseuille law) for a circular tube of height e filled with a mixture of porosity $\omega = \frac{V_p}{V}$ (V: total volume, $V_p = V - V_s$: volume of pores, μ viscosity)

$$\vec{U} = -\frac{\omega e^2}{12\mu}\nabla(p + \rho g z) := -K_*(\omega, \rho)\nabla h.$$

General approach: homogenization.

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General approach: homogenization.

Moisture and mass conservation

Case of unsaturated media: all the pores are not filled with water.
Water volume V_w. Moisture $\theta := \frac{V_w}{V} \in (0, \omega)$.
Darcy's law: $\vec{U} = -K(\theta)\nabla h$. (No attempt to derive it)
Conservation of mass for the water (q flux of water):

$$\partial_t(\rho\theta) + \text{div}(\rho\vec{U}) + \rho q = 0$$

Richards' equation

$$\partial_t \theta = \text{div}(K(\theta)\nabla h).$$

with a retention law: $\theta := \theta(h)$ such that $\theta(h) = \theta_s$ for $h \geq h_s$, (saturated medium). Define $C(h) = \frac{d\theta}{dh}$. Equation

$$C(h)\partial_t h = \text{div}(K(\theta(h))\nabla h).$$

Degenerate ($C(h) = 0, h \geq h_s$) parabolic-elliptic equation.
Other models have degeneracies (Caro, Saad,Saad, Apr. 2014) but assume $\theta(h) = \varphi(x)h$, $\varphi(x) \geq \varphi_1$ and the degeneracy is in the coupling term.

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Previous result (o.l., c.lp., 2002)

This result relies on a paper of Benilan and Wittbold. The functions K and C are continuous.

The system $\partial_t(\theta(h)) = \mathbf{div}(K(h)\partial_x h), (x,t) \in [0,1] \times R_+$, **inhomogeneous b.c. on** $h(x,t)$, $h(x,0) = h_i(x) \in W^{2,1}([0,1])$ **with** $\theta(h_i(.)) \in L^1([0,1])$ **has a unique solution in** $W^{1,\infty}([0,1], L^1([0,1]))$.

Note that, if $g_i(x)$ satisfies $\theta(g_i(.)) = \theta(h_i(.))$, then the solution is the same. The solution is **not** one to one w. r. to the initial condition.

Numerical scheme: $\theta(h_n) := \theta(h_{n+1}) + \Delta t \mathrm{div}(K(h_{n+1})\nabla h_{n+1})$. One gets h_{n+1} uniquely, and $\theta(h_n)$ converges. Badly conditioned for θ' and Δt small. Coercivity needed (C. LP).

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Some retention laws

$\Theta = \frac{\theta - \theta_{res}}{\theta_{sat} - \theta_{res}}$, $\psi = p_{air} - p_w$, hydrostatic pressure:

- Brooks and Corey (1964) $\Theta = (\frac{\psi_{ea}}{\psi})^{\lambda}$,
- Williams (1983) $\ln \Theta = A = B \ln \psi$,
- Van Genutchen (1980): $\Theta = (1 + (\alpha \Psi)^n)^{-m}$.

Analytic solution in the case $\theta(h) = min(h, 0)$ (C. Baudry, PhD), $h_\infty < 0 \leq h_0$:

$$\partial_t(\theta(h(x,t))) = D\frac{\partial^2 h}{\partial x^2}, h(x,0) = h_\infty 1_{x<0}, h(0,t) = h_0, t > 0,$$

$$h(x,t) = \begin{cases} h_0(1 - \frac{x}{2a\sqrt{t}}), x < 2a\sqrt{t} \\ h_\infty(1 - \frac{erfc(\frac{x}{2\sqrt{Dt}})}{erfc(a)}), x \geq 2a\sqrt{t} \end{cases}$$

where a solves $\frac{2}{\sqrt{\pi}} \frac{aexp(-a^2)}{erfc(a)} = -\frac{h_0}{h_\infty}$.

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Coupling with a mechanical model

From the literature (Green,Wang, Water res. res. 26 (7), 1990):

Biot's law (deformation tensor $\epsilon_{ij} = \frac{1}{2}(\partial_i X_j + \partial_j X_i)$)

$$2G\epsilon_{ij} = \sigma_{ij} + (\frac{2G}{3}(\frac{1}{K} - \frac{1}{K_s})p - \frac{1}{3}(1 - \frac{2G}{3K})(Tr(\sigma)))\delta_{ij}$$

equivalent to (note that $Tr(\epsilon) = \text{div}\vec{X}$)

$$\sigma_{ij} = 2G\epsilon_{ij} + (\alpha p + \beta Tr(\epsilon))\delta_{ij}.$$

Retention law:

$$\theta(h) = S(h)\omega = S(h)(\omega_0 + \frac{Tr(\epsilon)}{3}),$$

Relation between p and h: $p = \rho_w gh + p_{air}$,

Equilibrium (mechanical constraints)

$$\text{div}(\sigma - \rho_w ghId) = f$$

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Other models

- Old model (O.L., C. LP): $\epsilon^*_{ij} = \epsilon_{ij} + h_s p \delta_{ij} - h_s p_a \delta_{ij}$

$\sigma = \bar{\bar{D}}(\epsilon^*) \to \epsilon_{ij} = (\bar{\bar{T}}\sigma)_{ij} - \frac{\nu}{E} Tr(\epsilon)\delta_{ij} - h^s p \delta_{ij},$

- Hyperelastic model (Callari, Abati, 2011, eq. 14):

$$\partial_t \sigma = \bar{\bar{C}} \partial_t \epsilon - \partial_t p b$$

All three models have in common that $\partial_t \epsilon = \bar{\bar{D}} \partial_t \sigma - \partial_t h V^s Id.$

$$\Rightarrow \partial_t \omega = \frac{1}{3} \text{div}(\bar{\bar{D}} \partial_t \sigma) - \frac{V_1^s + V_2^s + V_3^s}{3} \partial_t h.$$

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Expression of the deformation rate $\omega = \frac{1}{3}Tr(\epsilon)$

Equation (equilibrium):

$$-\text{div}(2G\epsilon + \beta Tr(\epsilon)Id) = \rho_w g(\alpha - 1)\nabla h - f + \alpha\nabla p_a$$

Solved on Ω bounded, regular with a inhomogeneous Dirichlet boundary condition on $\vec{X}$ on $\Gamma \subset \partial\Omega$: the operator K_T

$$(X_1, X_2, X_3) \to -2G\text{div}\epsilon - \beta\nabla(Tr(\epsilon))$$

is self-adjoint coercive and continuous from $(H_0^1(\Omega))^3$ to $(H^{-1}(\Omega))^3 = ((H_0^1(\Omega))')^3$,

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$$S'(h)\omega(h)\partial_t h + S(h)L(\partial_t h) = \mathrm{div}(K(\theta)\nabla h) + S(h)\partial_t F$$

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Operator on the right hand side still coercive in $H^1 \to$ regularized problem on $\tilde{\theta}$.

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Conclusions

- Mechanical behavior (hyperelastic, elastic): $\epsilon = D\sigma - hVId$
- Assumption $\alpha = \frac{1}{\rho_w g} D^{-1} V < 1$:$\Rightarrow \vec{X} = C_0 K_T^{-1} \nabla h + S_i$, $C_0 > 0, K_T$ coercive, which imply, from $\theta(h) = S(h)\omega$, $S'(h) \geq 0$ and $S(h) > 0$, (with suitable b.c. for $\vec{X}$ or $\dot{\vec{X}}$), that

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- Already observed in some numerical resolutions coupling the models (c.Ip for example) where one includes in the system an additional term in $(S'(h)\omega + \tilde{c})\partial_t h = \text{div}(K(h)\nabla h)$

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