NXFEM for solving non-standard transmission problems

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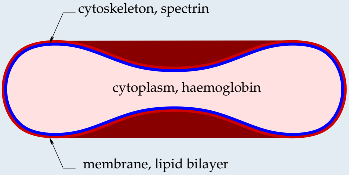
Motivation and goal

Motivation

- Interaction between immiscible fluids (Newtonian and non-Newtonian) but also porous media, involving thin layers

Applications:

- biological liquids, e.g. red blood cells (PhD. thesis of H. El-Otmany)
- flows in fractured porous media



Our goal

- Asymptotic modeling
- Numerical treatment of interfaces
- Conforming but also nonconforming finite elements:
  - inf-sup stable for Stokes equations
  - well-adapted to treat thin layers (no numerical locking ...)

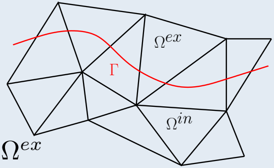
$\Longrightarrow$ **Development of NXFEM for non-standard interface conditions**

Outline

1. Presentation of NXFEM

Model problem

$$\begin{cases} -\nabla \cdot (\mu \nabla u) = f & \text{in } \Omega^{in} \cup \Omega^{ex} \\ u = 0 & \text{on } \partial\Omega \\ [u] = 0 & \text{on } \Gamma \\ [\mu \nabla_n u] = g & \text{on } \Gamma \end{cases}$$



where $[u] = u^{in} - u^{ex}$, $\mu = \mu^{in}$ in Ω^{in}, $\mu = \mu^{ex}$ in Ω^{ex}

General idea of NXFEM

- designed to take into account discontinuities on non-aligned meshes
- introduced for conforming approximations of elliptic problems (cf. Hansbo & Hansbo, *CMAME 2002*)

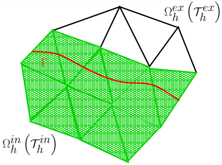
Characteristics:

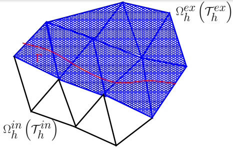
- variational problem with standard FE spaces enriched on cut cells
- interface conditions treated weakly, by Nitsche's method

Discrete variational formulation

$$u_h \in W_h^{in} \times W_h^{ex}, \quad a_h(u_h, v_h) = l(v_h), \quad \forall v_h \in W_h^{in} \times W_h^{ex}$$

$$W_h^i := \left\{ \varphi \in H^1(\Omega_h^i); \varphi|_T \in P^1, \forall T \in \mathcal{T}_h^i, \varphi|_{\partial\Omega} = 0 \right\}, \quad i = in, ex$$





Bilinear and linear forms

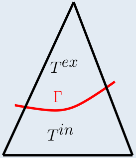
$$a_h(u_h, v_h) := \sum_{T \in \mathcal{T}_h} \int_T \mu \nabla u_h \cdot \nabla v_h - \int_\Gamma \{\mu \nabla_n u_h\} [v_h] - \int_\Gamma \{\mu \nabla_n v_h\} [u_h]$$

$$+ \xi \sum_{T \in \mathcal{T}_h^\Gamma} \gamma_T \int_{\Gamma_T} [u_h] [v_h]$$

$$l(v_h) := \int_\Omega f v_h + \int_\Gamma g \{v_h\}_*$$

Weighted means and choice of parameters

$$\{u\} = k^{ex}u^{ex} + k^{in}u^{in}$$
$$\{u\}_* = k^{in}u^{ex} + k^{ex}u^{in}$$
$$k^{in} + k^{ex} = 1, \quad 0 < k^{in},\, k^{ex} < 1$$



- originally (Hansbo & Hansbo, *CMAME 2002*):

$$k^{in} = \frac{|T^{in}|}{|T|}, \quad k^{ex} = \frac{|T^{ex}|}{|T|}, \quad \gamma_T = \frac{4\max\left(\mu^{in}, \mu^{ex}\right)}{|T|}$$

$\rightsquigarrow$ robustness with respect to the mesh-interface geometry.

- improvement (Barrau, Becker, Dubach & Luce, *CRAS 2012*):

$$k^{in} = \frac{\mu^{ex}|T^{in}|}{\mu^{ex}|T^{in}| + \mu^{in}|T^{ex}|}, \; k^{ex} = \frac{\mu^{in}|T^{ex}|}{\mu^{ex}|T^{in}| + \mu^{in}|T^{ex}|}, \; \gamma_T = \frac{\mu^{in}\mu^{ex}|T|}{\mu^{in}|T^{ex}| + \mu^{ex}|T^{in}|}$$

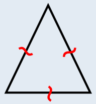
(see also Annavarapu, Hautefeuille, Dolbow, *CMAME 2012*)

$\rightsquigarrow$ robustness with respect to the diffusion parameters, too.

2. Extension of NXFEM to nonconforming FE

Nonconforming Crouzeix-Raviart elements

DOF: $\frac{1}{|e_i|}\int_{e_i} v ds, \quad 1 \leq i \leq 3$



Notation

- $\mathcal{T}_h^\Gamma$: set of cells cut by Γ
- $\mathcal{E}_h^i$: set of edges of $\mathcal{T}_h^i$
- $\mathcal{E}_h^{i,cut}$: set of cut edges contained in Ω^i
- $\mathcal{N}_h^\Gamma$: intersection between cut edges and Γ

Difficulty of the extension

- Conforming case: interpolation operator $I_h^* = \left(I_h^{*in}, I_h^{*ex}\right)$ on $W_h^{in} \times W_h^{ex}$

$$v|_{\Omega^i} \longrightarrow E^i v|_{\Omega} \longrightarrow I_h\left(E^i v\right)|_{\Omega} \longrightarrow I_h\left(E^i v\right)|_{\Omega_h^i} =: I_h^{*i} v$$

- Nonconforming case: W_h^i replaced by

$$V_h^i = \left\{\varphi \in L^2(\Omega_h^i);\ \varphi|_T \in P^1,\ \forall T \in \mathcal{T}_h^i,\ \int_e [\varphi] = 0,\ \forall e \in \mathcal{E}_h^i\right\}$$

Then

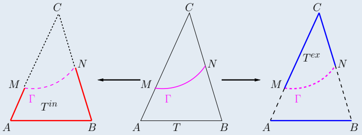
$$\int_e I_h^{*i} v \neq \int_e v, \quad \forall e \in \mathcal{E}_h^{i,cut} \qquad i = in, ex$$

$\rightsquigarrow$ problem to estimate the consistency error on the cut edges

$$\sum_{i=in,\,ex} \sum_{e \in \mathcal{E}_h^{i,cut}} \int_e \mu^i \nabla_n u^i [v_h^i]$$

Proposed solutions (PhD. thesis of H. El-Otmany)

- Modification of the basis functions on cut cells (DOF on cut edges)



$$u_h \in \tilde{V}_h, \quad a_h(u_h, v_h) = l(v_h), \quad \forall v_h \in \tilde{V}_h$$

- Addition of stabilization terms on cut edges

$$u_h^\delta \in V_h, \quad \left(a_h + A_h + \sum_{i=in,ex} \delta^i J_h^i\right)\left(u_h^\delta, v_h\right) = l\left(v_h\right), \quad \forall v_h \in V_h$$

- Relationship between the two approaches

$$\lim_{\delta^i \longrightarrow +\infty} |||u_h^\delta - u_h||| = 0$$

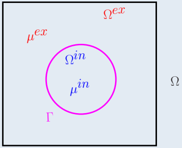
Generalization to Stokes equations

Numerical test

Reference test-case (Hansbo & Hansbo '02)

Data:

- $\Omega = (0,1)^2$, $r = \sqrt{x^2+y^2}$, $r_0 = 3/4$
- $\mu^{in} = 1$, $\mu^{ex} = 10^3$



Stabilization parameters:

- $\xi = 10$, $\delta^{in} = \delta^{ex} = 100$

The exact solution is given by:

$$u(x,y) = \begin{cases} \dfrac{r^2}{\mu^{in}} & \text{if } r \le r_0 \\ \dfrac{r^2}{\mu^{ex}} - \dfrac{r_0^2}{\mu^{ex}} + \dfrac{r_0^2}{\mu^{in}} & \text{if } r > r_0, \end{cases}$$

Implementation in the C++ library CONCHA

- conforming case (PhD. thesis of N. Barrau)
- nonconforming case: second approach

Comparison with conforming NXFEM

Conforming FE

N	energy norm	ratio	L^2-norm	ratio
64	3.45e-01	1.00	2.83e-02	1.00
256	1.68e-01	2.05	6.27e-03	4.52
1024	8.03e-02	2.09	1.41e-03	4.45
4096	3.95e-02	2.03	3.38e-04	4.17
16384	1.97e-02	2.01	8.21e-05	4.11
65536	9.82e-03	2.00	2.02e-05	4.06

Nonconforming FE with stabilization

N	energy norm	ratio	L^2-norm	ratio
64	3.93e-01	1.00	3.16e-02	1.00
256	1.66e-01	2.36	6.03e-03	4.24
1024	7.89e-02	2.11	1.36e-03	4.44
4096	3.88e-02	2.03	3.24e-04	4.19
16384	1.88e-02	2.07	7.69e-05	4.21
65536	9.73e-03	2.05	1.71e-05	4.13

Nonconforming NXFEM: computed solution

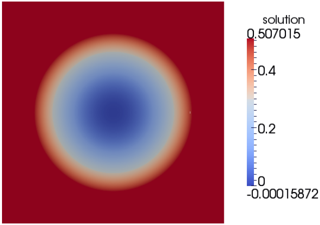


Fig: Exact solution

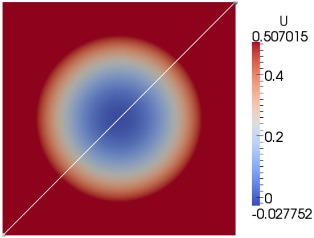


Fig: Computed solution ($N = 65536$)

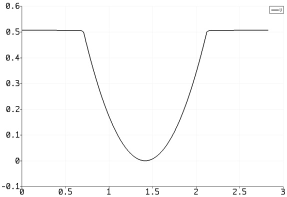
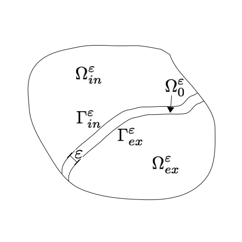


Fig: Profile along a diagonal

3. Darcy flow with a thin layer

Model problem

$$\begin{cases} -\nabla\cdot(K\nabla\tilde{u}_\varepsilon) &= f \quad \text{in } \Omega_\varepsilon^{in}\cup\Omega_\varepsilon^{0}\cup\Omega_\varepsilon^{ex} \\ \tilde{u}_\varepsilon &= 0 \quad \text{on } \partial\Omega_\varepsilon \\ [\tilde{u}_\varepsilon] &= 0 \quad \text{on } \Gamma_\varepsilon^{in}\cup\Gamma_\varepsilon^{ex} \\ [K\nabla\tilde{u}_\varepsilon\cdot n] &= 0 \quad \text{on } \Gamma_\varepsilon^{in}\cup\Gamma_\varepsilon^{ex} \end{cases}$$



where K is a symmetric, positive definite tensor and

$$K = \begin{cases} K^{in} & \text{in } \Omega_\varepsilon^{in} \\ K_\varepsilon^0 & \text{in } \Omega_\varepsilon^{0} \\ K^{ex} & \text{in } \Omega_\varepsilon^{ex} \end{cases}, \quad f = \begin{cases} f^{in} & \text{in } \Omega_\varepsilon^{in} \\ 0 & \text{in } \Omega_\varepsilon^{0} \\ f^{ex} & \text{in } \Omega_\varepsilon^{ex} \end{cases}, \quad K^0 = \lim_{\varepsilon\to 0}\varepsilon K_\varepsilon^0$$

$$\Omega_\varepsilon^0 = \left\{\zeta\in\mathbb{R}^2;\zeta=\xi+\varepsilon l n(\xi),\ \xi\in\Gamma \text{ and } -\frac{h(\xi)}{2}<l<\frac{h(\xi)}{2}\right\}$$

n unit normal to the mean curve Γ, $h:\bar{\Gamma}\longrightarrow\mathbb{R}$ smooth and bounded

Asymptotic modeling

Assumptions: rectilinear mean curve ($\Gamma = [0,1]$), constant thickness ($h = 1$).

Change of variables

$$(x,y) \in \Omega_\varepsilon^i \rightsquigarrow (s,l) \in \Omega^i, \qquad \tilde{v}(x,y) = v(s,l)$$

Domain (x,y)	New domain (s,l)	Change of variables
$\Omega_\varepsilon^{in} =]0,1[\times]-1-\frac{\varepsilon}{2}, \frac{-\varepsilon}{2}[$	$\Omega^{in} =]0,1[\times]\frac{-3}{2}, \frac{-1}{2}[$	$s = x, \quad l = y + \frac{\varepsilon - 1}{2}$ $\nabla_{s,l} v = \nabla_{x,y} \tilde{v}$
$\Omega_\varepsilon^{0} =]0,1[\times]\frac{-\varepsilon}{2}, \frac{\varepsilon}{2}[$	$\Omega^{0} =]0,1[\times]\frac{-1}{2}, \frac{1}{2}[$	$s = x, \quad l = \frac{1}{\varepsilon} y$ $\partial_s v = \partial_x \tilde{v}, \quad \partial_l v = \varepsilon \partial_y \tilde{v}$
$\Omega_\varepsilon^{ex} =]0,1[\times]\frac{\varepsilon}{2}, 1+\frac{\varepsilon}{2}[$	$\Omega^{ex} =]0,1[\times]\frac{1}{2}, \frac{3}{2}[$	$s = x, \quad l = y - \frac{\varepsilon - 1}{2}$ $\nabla_{s,l} v = \nabla_{x,y} \tilde{v}$

Weak formulation

Variational problem

$$u_\varepsilon \in V, \quad a_\varepsilon(u_\varepsilon, v) = \int_{\Omega^{in} \cup \Omega^{ex}} fv, \quad \forall v \in V$$

where:

$$a_\varepsilon(u,v) = \int_{\Omega^{in}} K^{in} \nabla u \cdot \nabla v + \int_{\Omega^{ex}} K^{ex} \nabla u \cdot \nabla v +$$
$$\int_{\Omega^0} \left(\varepsilon K^0_{\varepsilon,11} \partial_s u \partial_s v + K^0_{\varepsilon,12} (\partial_s u \partial_l v + \partial_l u \partial_s v) + \frac{1}{\varepsilon} K^0_{\varepsilon,22} \partial_l u \partial_l v \right)$$
$$V = H^1_0(\Omega)$$

Key point

Uniform coercivity for ε small enough: $\exists c > 0$ s.t.

$$a_\varepsilon(v,v) \geq c \|v\|^2_V, \quad \forall v \in V$$

Convergence of u_ε as $\varepsilon \longrightarrow 0$

- $\|u_\varepsilon\|_V \le c\|f\|_{0,\Omega^{in}\cup\Omega^{ex}}$ and $\|\partial_l u_\varepsilon\|_{0,\Omega^0} \le c\,\varepsilon\|f\|_{0,\Omega^{in}\cup\Omega^{ex}}$

  $\Longrightarrow$ $u_\varepsilon \rightharpoonup u_0$ in V (at least a subsequence)

  $\partial_l u_\varepsilon \to 0$ in $L^2(\Omega^0)$ and $\partial_l u_0 = 0$ a.e. in Ω^0

  $\frac{1}{\varepsilon}\partial_l u_\varepsilon \rightharpoonup \omega_0$ in $L^2(\Omega^0)$ (at least a subsequence)

- $\displaystyle\int_{\Omega^0} \left(K^0_{12}\partial_s u_0 + K^0_{22}\omega_0\right)\partial_l v = 0,\ \forall v \in V$

  $\Longrightarrow$ $\omega_0(s,l) = -\dfrac{K^0_{12}}{K^0_{22}}\partial_s u_0$ a.e. in Ω^0

  If K^0_ε is diagonal, then $\omega_0 = 0$.

Limit problem

Variational limit problem

$$u_0 \in V_0, \quad a_0(u_0, v) = \int_{\Omega^{in} \cup \Omega^{ex}} f v, \quad \forall v \in V_0$$

where:

$$a_0(u,v) = \int_{\Omega^{in}} K^{in} \nabla u \cdot \nabla v + \int_{\Omega^{ex}} K^{ex} \nabla u \cdot \nabla v + \int_\Gamma \alpha_0(s)\, \partial_s u\, \partial_s v$$

$$\alpha_0(s) = \int_{-1/2}^{1/2} \frac{\det K^0(s,l)}{K^0_{22}(s,l)} dl$$

$$V_0 = \left\{ v \in V;\ \partial_l v = 0 \text{ in } \Omega^0 \right\}$$

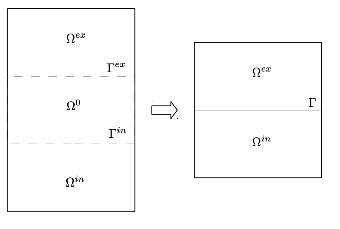
- Well-posed problem w.r.t. $|||v|||^2 = \sum_{i=in,ex} \|(K^i)^{1/2} \nabla v\|^2_{0,\Omega^i} + \|\alpha_0^{1/2} \partial_s v\|^2_{0,\Gamma}$

$$\Longrightarrow \quad u_\varepsilon \to u_0 \ \text{ in } \ V \ \text{(the whole sequence)}$$

Limit problem

Asymptotic model problem

Let $\Gamma^{in}, \Gamma^{ex} \rightsquigarrow \Gamma, \quad \Omega^{in} \rightsquigarrow \Gamma \times]-1,0[, \quad \Omega^{ex} \rightsquigarrow \Gamma \times]0,1[, \quad \Omega =]0,1[\times]-1,1[$



$$
\left\{
\begin{array}{rcll}
-\nabla \cdot (K \nabla u_0) & = & f & \text{in } \Omega^{in} \cup \Omega^{ex} \\
u_0 & = & 0 & \text{on } \partial\Omega \\
[u_0] & = & 0 & \text{on } \Gamma \\
[K \nabla u_0 \cdot n] - \partial_s(\alpha_0 \partial_s u_0) & = & 0 & \text{on } \Gamma
\end{array}
\right.
$$

For a single domain with a thin layer: Ventcel's boundary condition.

Extension to a smooth curved interface

Curvilinear coordinates

$$\Omega_\varepsilon^0 = \left\{ \zeta \in \mathbb{R}^2; \zeta = \xi + \varepsilon l n(\xi),\, \xi \in \Gamma \text{ and } -\frac{h(\xi)}{2} < l < \frac{h(\xi)}{2} \right\}$$

Assume the thickness is constant $(h = 1)$ and the mean curve Γ is smooth:

$$s \in [0,1] \rightarrow \xi = \xi(s) \in \Gamma, \quad s \text{ curvilinear abscissa}, \quad \{\tau, n\} \text{ Frenet basis}$$

Frenet's formulae: $\dfrac{d\tau}{ds} = r\, n, \quad \dfrac{dn}{ds} = -r\, \tau$ with $r = r(s)$ the curvature of Γ

In curvilinear coordinates (s, l), one has $\nabla u = \left(\dfrac{\partial_s u}{1 - \varepsilon\, r l}, \dfrac{\partial_l u}{\varepsilon} \right)^T$.

Finally, we get the same interface condition on Γ:

$$[K \nabla u_0 \cdot n] - \partial_\tau(\alpha_0 \partial_\tau u_0) = 0$$

NXFEM for the asymptotic model

Goal

Development of a stable and consistent numerical method of NXFEM type

We focus on conforming FE (... natural extension to nonconforming FE).

For the sake of simplicity, assume K^i diagonal: $K^i = \kappa^i I$. Then $\alpha_0 = \kappa^0$.

Consistency

$$a_h(u, v_h) - l(v_h) = \int_\Gamma \partial_s(\alpha_0 \partial_s u)\{v_h\}_*$$

$$= - \sum_{T \in \mathcal{T}_h^\Gamma} \int_{\Gamma_T} \alpha_0 \, \partial_s u \, \partial_s \{v_h\}_* + \sum_{N \in \mathcal{N}_h^\Gamma} (\alpha_0 \partial_s u)(N) [\{v_h\}_*]_N$$

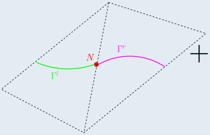
- $\{v_h\}_*$ is discontinuous along Γ due to the weights k^{in}, k^{ex} in the mean
- $[u] = 0$ implies $\partial_s u = \{\partial_s u\} = \{\partial_s u\}_*$ on Γ
- $\partial_s \{\varphi\}_* = \{\partial_s \varphi\}_*$ on Γ_T because k^{in}, k^{ex} are constant on Γ_T

NXFEM for the asymptotic model

Bilinear form

$$a_h^{new}(u_h, v_h) := a_h(u_h, v_h) + \sum_{T \in \mathcal{T}_h^\Gamma} \int_{\Gamma_T} \alpha_0 \, \partial_s \{u_h\}_* \, \partial_s \{v_h\}_*$$

$$- \sum_{N \in \mathcal{N}_h^\Gamma} \alpha_0(N) \Big(\{\partial_s \{u_h\}_*\}_N [\{v_h\}_*]_N + \{\partial_s \{v_h\}_*\}_N [\{u_h\}_*]_N \Big)$$



$$+ \gamma \sum_{N \in \mathcal{N}_h^\Gamma} \gamma_N [\{u_h\}_*]_N [\{v_h\}_*]_N$$

with $\gamma > 0$, $\gamma_N := \dfrac{\alpha_0(N)}{|\Gamma^l| + |\Gamma^r|}$ and the jump / mean at a node $N \in \mathcal{N}_h^\Gamma$:

$$[\varphi]_N := \varphi^l - \varphi^r, \quad \{\varphi\}_N := \nu^l \varphi^l + \nu^r \varphi^r, \quad \nu^l = \frac{|\Gamma^l|}{|\Gamma^l| + |\Gamma^r|}, \quad \nu^r = \frac{|\Gamma^r|}{|\Gamma^l| + |\Gamma^r|}$$

Since $[\{u\}_*]_N = [u]_N = 0$ and $\{\partial_s \{u\}_*\}_N = \partial_s u(N)$, consistency follows:

$$a_h^{new}(u, v_h) - l(v_h) = 0, \quad \forall v_h \in W_h^{in} \times W_h^{ex}$$

NXFEM for the asymptotic model

Stability

$$|||\varphi|||^2 = \sum_{i=in,ex} \|K^{1/2}\nabla\varphi\|^2_{0,\Omega^i} + \sum_{T\in\mathcal{T}_h^\Gamma} h_T \|\{K\nabla_n\varphi\}\|^2_{0,\Gamma^T} + \sum_{T\in\mathcal{T}_h^\Gamma} \gamma_T \|[\varphi]\|^2_{0,\Gamma^T}$$

$$\|\varphi\|^2_{new} = |||\varphi|||^2 + \sum_{T\in\mathcal{T}_h^\Gamma} \|\alpha_0^{1/2}\partial_s\{\varphi\}_*\|^2_{0,\Gamma^T} + \sum_{N\in\mathcal{N}_h^\Gamma} \gamma_N [\{\varphi\}_*]^2_N$$

Thanks to the choice of ν^l, ν^r in $\{\cdot\}_N$ and to $\partial_s\{v_h\}_*$ constant on Γ^l, Γ^r $\Longrightarrow$

$$\{\partial_s\{v_h\}_*\}^2_N \leq \frac{1}{|\Gamma^l| + |\Gamma^r|}\left(\|\partial_s\{v_h\}_*\|^2_{0,\Gamma^l} + \|\partial_s\{v_h\}_*\|^2_{0,\Gamma^r}\right)$$

For simplicity, assume α_0 constant. Then

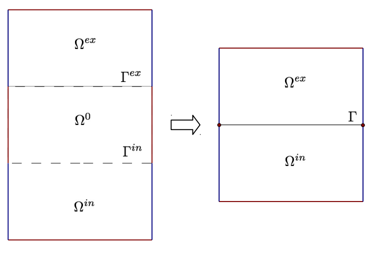
$$|\alpha_0(N)\{\partial_s\{u_h\}_*\}_N[\{v_h\}_*]_N| \leq \|\alpha_0^{1/2}\partial_s\{u_h\}_*\|_{0,\Gamma^l\cup\Gamma^r}\left(\gamma_N^{1/2}|[\{v_h\}_*]_N|\right)$$

For ξ and γ large enough, stability follows:

$$a_h^{new}(v_h, v_h) \geq c\|v_h\|^2_{new}, \quad \forall v_h \in W_h^{in} \times W_h^{ex}$$

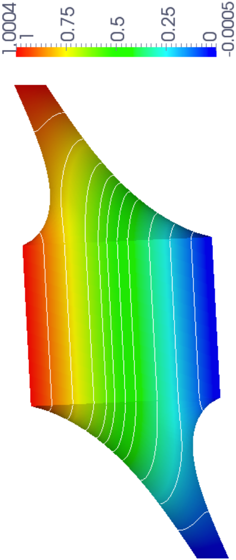
Numerical test

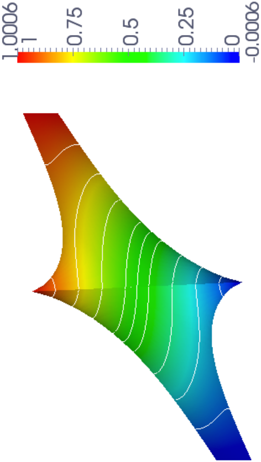
- $f = 0$, $K \frac{\partial u}{\partial n} = 0$ on Γ_N, $u = u_D$ on Γ_D
- $\kappa^{in} = \kappa^{ex} = 1$, $\kappa^0 = 2000$, $\varepsilon = 0.001$



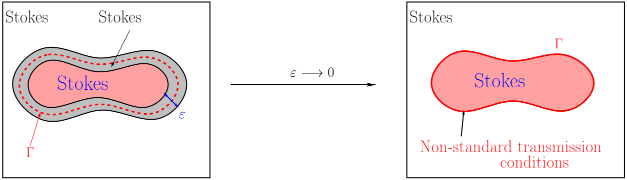
- same test as in Frih, Martin, Roberts & Saada, *Comput. Geosci. 2012* (different limit problem; interface aligned with the mesh)
- we obtain similar numerical results (both for aligned and not aligned meshes)

Comparison between u_ε and u_0





4. Stokes flow with a thin layer



Original model problem

Limit model problem

Model problem

$$\begin{cases} -\mu\Delta\tilde{u}_\varepsilon + \nabla\tilde{p}_\varepsilon = f & \text{in } \Omega_\varepsilon^{in}\cup\Omega_\varepsilon^{0}\cup\Omega_\varepsilon^{ex} \\ \operatorname{div}\tilde{u}_\varepsilon = 0 & \text{in } \Omega_\varepsilon^{in}\cup\Omega_\varepsilon^{0}\cup\Omega_\varepsilon^{ex} \\ \tilde{u}_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon \\ [\tilde{u}_\varepsilon] = 0 & \text{on } \Gamma_\varepsilon^{in}\cup\Gamma_\varepsilon^{ex} \\ [\mu\partial_n\tilde{u}_\varepsilon - \tilde{p}_\varepsilon n] = g & \text{on } \Gamma_\varepsilon^{in}\cup\Gamma_\varepsilon^{ex} \end{cases}$$

where $\mu = \begin{cases} \mu^{in} & \text{in } \Omega_\varepsilon^{in}, \\ \dfrac{\mu^0}{\varepsilon} & \text{in } \Omega_\varepsilon^{0}, \\ \mu^{ex} & \text{in } \Omega_\varepsilon^{ex}, \end{cases}$ $\quad f = \begin{cases} f^{in} & \text{in } \Omega_\varepsilon^{in} \\ \dfrac{f^0}{\varepsilon} & \text{in } \Omega_\varepsilon^{0} \\ f^{ex} & \text{in } \Omega_\varepsilon^{ex} \end{cases}$ $\quad g = \begin{cases} g^{in} & \text{on } \Gamma_\varepsilon^{in} \\ g^{ex} & \text{on } \Gamma_\varepsilon^{ex} \end{cases}$

Rectilinear interface

Change of variables

$$(x, y) \in \Omega_\varepsilon^i \rightsquigarrow (s, l) \in \Omega^i, \qquad \tilde{v}(x, y) = v(s, l)$$

In the thin layer:

$$\partial_x \tilde{v} = \partial_s v, \quad \partial_y \tilde{v} = \frac{1}{\varepsilon}\partial_l v, \quad \mathrm{div}_{x,y}(\tilde{v}_1, \tilde{v}_2)^T = \partial_s v_1 + \frac{1}{\varepsilon}\partial_l v_2, \quad dx\, dy = \varepsilon\, ds\, dl$$

Change of unknown pressure in Ω^0 : $\qquad \varepsilon p_\varepsilon^0 \rightsquigarrow p_\varepsilon^0$

Mixed formulation

Mixed weak formulation

$$(u_\varepsilon, p_\varepsilon) \in H_0^1(\Omega) \times L_0^2(\Omega)$$

$$\begin{cases} a_\varepsilon(u_\varepsilon, v) - b_\varepsilon(p_\varepsilon, v) = & L(v), \quad \forall v \in H_0^1(\Omega) \\ b_\varepsilon(q, u_\varepsilon) = & 0, \quad \forall q \in L_0^2(\Omega) \end{cases}$$

$$\begin{aligned} a_\varepsilon(u,v) &= \sum_{i=in,ex} \int_{\Omega^i} \mu^i \nabla u : \nabla v + \int_{\Omega^0} \mu^0 \partial_s u \cdot \partial_s v + \frac{1}{\varepsilon^2} \int_{\Omega^0} \mu^0 \partial_l u \cdot \partial_l v \\ &= a(u,v) + \frac{1}{\varepsilon^2} a_0(u,v) \\ b_\varepsilon(p,v) &= \sum_{i=in,ex} \int_{\Omega^i} p \,\mathrm{div} v + \int_{\Omega^0} p\, \partial_s v_1 + \frac{1}{\varepsilon} \int_{\Omega^0} p\, \partial_l v_2 \\ &= b(p,v) + \frac{1}{\varepsilon} b_0(p,v) \\ L(v) &= \sum_{i=in,0,ex} \int_{\Omega^i} f^i \cdot v + \sum_{i=in,ex} \int_{\Gamma^i} g^i \cdot v \end{aligned}$$

Convergence of $(u_\varepsilon, p_\varepsilon)$ towards (u_0, p_0)

Uniform well-posedness of mixed formulation

- $\|v\|_V^2 = \sum_{i=in,0,ex} \|\mu_i^{1/2}\nabla v^i\|_{0,\Omega^i}^2, \quad \|q\|_M^2 = \sum_{i=in,0,ex} \|\mu_i^{-1/2} q^i\|_{0,\Omega^i}^2$
- uniform coercivity of $a_\varepsilon(\cdot,\cdot)$ in $H_0^1(\Omega)$

  $\Longrightarrow$ $u_\varepsilon \rightharpoonup u_0$ in $H_0^1(\Omega)$ (at least a subsequence), $\partial_l u_\varepsilon \to 0$ in $L^2(\Omega^0)$

$$u_0 \in \operatorname{Ker} a_0 = \left\{ v \in H_0^1(\Omega);\, \partial_l v = 0 \text{ in } \Omega^0 \right\} =: V_0$$

- key point: inf-sup condition of $b(\cdot,\cdot)$ on $M_0 \times V_0$ where

$$M_0 := \left\{ q \in L_0^2(\Omega);\, q = q(s) \text{ in } \Omega^0 \right\}$$

Let $\hat{p}_\varepsilon^0(s) = \int_{-1/2}^{1/2} p_\varepsilon^0(s,l)\, dl$ for $s \in \Gamma$. Then $\hat{p}_\varepsilon := (p_\varepsilon^{in}, \hat{p}_\varepsilon^0, p_\varepsilon^{ex}) \in M_0$ and

$$\|\hat{p}_\varepsilon\|_M \le \frac{1}{\beta} \sup_{v \in V_0} \frac{b(\hat{p}_\varepsilon, v)}{\|v\|_V} = \frac{1}{\beta} \sup_{v \in V_0} \frac{a(u_\varepsilon, v) - L(v)}{\|v\|_V} \le C \|u_\varepsilon\|_V$$

$\Longrightarrow$ $\hat{p}_\varepsilon \rightharpoonup p_0$ in $L^2(\Omega)$ (at least a subsequence), $\quad p_0 \in M_0$

Limit problem

Variational limit problem

$$(u_0, p_0) \in V_0 \times M_0$$

$$\begin{cases} a(u_0, v) - b(p_0, v) = & L(v), \quad \forall v \in V_0 \\ b(q, u_0) = & 0, \quad \forall q \in M_0 \end{cases}$$

- well-posed mixed problem (Babuska-Brezzi theorem)

$$\Longrightarrow \quad (u_\varepsilon, \hat{p}_\varepsilon) \to (u_0, p_0) \quad \text{(the whole sequence)}$$

- $\Gamma^{in}, \Gamma^{ex} \rightsquigarrow \Gamma, \quad \Omega^{in} \rightsquigarrow \Gamma \times]-1, 0[, \quad \Omega^{ex} \rightsquigarrow \Gamma \times]0, 1[, \quad \Omega \rightsquigarrow]0, 1[\times]-1, 1[$

$$V_0 \rightsquigarrow \left\{ v \in H_0^1(\Omega);\ v_{|\Gamma} \in H_0^1(\Gamma) \right\}$$
$$M_0 \rightsquigarrow \left\{ (q, q^\Gamma) \in L^2(\Omega) \times L^2(\Gamma);\ \int_\Omega q + \int_\Gamma q^\Gamma = 0 \right\}$$

- For $v \in V_0$, we denote $v^\Gamma := v_{|\Gamma}$

Limit problem

Asymptotic model problem

$$\begin{cases} -\mu\Delta u_0 + \nabla p_0 = f & \text{in } \Omega^{in} \cup \Omega^{ex} \\ \operatorname{div} u_0 = 0 & \text{in } \Omega^{in} \cup \Omega^{ex} \\ u_0 = 0 & \text{on } \partial\Omega \\ [u_0] = 0 & \text{on } \Gamma \\ u^{\Gamma}_{0,1} = 0 & \text{on } \Gamma \\ [\mu\partial_n u_0 - p_0 n] - \begin{pmatrix} -\partial_s p^{\Gamma}_0 \\ \partial_s(\mu^0 \partial_s u^{\Gamma}_{0,2}) \end{pmatrix} = \bar{f}^0 + g^{in} + g^{ex} & \text{on } \Gamma \end{cases}$$

Unknowns: (u^{in}_0, u^{ex}_0) and $(p^{in}_0, p^{ex}_0, p^{\Gamma}_0)$

Extension to a smooth curved interface

5. Perspectives

Extension

- Thin layer of non-Newtonian fluid
  - Newtonian constitutive law:
    $$\boldsymbol{\tau} = 2\eta \boldsymbol{D}, \qquad \boldsymbol{D} = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$$
  - quasi-linear constitutive law, popular but not realistic Oldroyd-B model:
    $$\boldsymbol{\tau} + \lambda_t \overset{\triangledown}{\boldsymbol{\tau}} = 2\eta\left(\boldsymbol{D} + \lambda_r \overset{\triangledown}{\boldsymbol{D}}\right), \qquad \overset{\triangledown}{\boldsymbol{A}} = \frac{\partial}{\partial t}\boldsymbol{A} + u \cdot \nabla \boldsymbol{A} - \boldsymbol{A}\nabla u^T - \nabla u \boldsymbol{A}$$
  - nonlinear constitutive law, more complex but realistic Giesekus model:
    $$\boldsymbol{\tau} + \lambda \overset{\triangledown}{\boldsymbol{\tau}} + \frac{\eta}{2\lambda}\boldsymbol{\tau} \cdot \boldsymbol{\tau} = 2\eta\, \boldsymbol{D}$$

Ongoing work

- Numerical method for Stokes equations with previous interface conditions

Future work

- Implementation and numerical validation
- Moving interface

Inf-sup condition on $M_0 \times V_0$

Steps of the proof

- For any $p = (p^{in}, p^0, p^{ex}) \in M_0$, let $\bar{p} = (\bar{p}^{in}, \bar{p}^0, \bar{p}^{ex})$ with $\bar{p}^i = \frac{1}{|\Omega^i|} \int_{\Omega^i} p^i$. Let $\tilde{p} = p - \bar{p}$. Then for any $v \in V_0$:

$$b(p,v) = \int_{\Omega^{in} \cup \Omega^{ex}} \tilde{p} \, \mathrm{div} v + \int_\Gamma \tilde{p}^0 \partial_s v_1 + \int_\Gamma (\bar{p}^{in} - \bar{p}^{ex}) \, v \cdot n$$

- $\tilde{p}^i \in L^2_0(\Omega^i)$ for $i = in$, ex, hence standard inf-sup condition for 1st term:

$$\exists \tilde{v}^i \in H^1_0(\Omega^i) \quad \text{s.t. } \tilde{v} = \left(\tilde{v}^{in}, 0, \tilde{v}^{ex}\right) \in V_0, \quad b(p, \tilde{v}) = \|\mu^{-1/2} \tilde{p}\|^2_{0,\Omega^{in} \cup \Omega^{ex}}$$

- $\tilde{p}^0 \in L^2_0(\Gamma)$ so $\exists \tilde{v}^0 \in H^1_0(\Gamma)$ and a continuous extension $v^\Gamma \in V_0$ s.t.

$$b(p, v^\Gamma) = \|\mu_0^{-1/2} \tilde{p}^0\|^2_{0,\Omega^0} + \int_{\Omega^{in} \cup \Omega^{ex}} p \, \mathrm{div} \, v^\Gamma$$

- for $\bar{p}^{in} - \bar{p}^{ex} \in \mathbb{R}$ there exists $\bar{v} \in V_0$ s.t.

$$\int_\Gamma (\bar{p}^{in} - \bar{p}^{ex}) \bar{v} \cdot n = \left(\bar{p}^{in} - \bar{p}^{ex}\right)^2 \geq C \|\bar{p}\|^2_M$$

Inf-sup condition on $M_0 \times V_0$

Steps of the proof

- take $v = \alpha\tilde{v} + \beta v^\Gamma + \gamma\bar{v}$ with $\alpha, \beta, \gamma > 0$ chosen by using Young's inequality s.t.

$$b(p, v) \geq c_1 \|p\|_M^2, \quad \|v\|_V \leq c_2 \|p\|_M$$