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Nonlinear Transport-Flow through Elastoviscoplastic Porous Media

Gonzalo Alduncin

Instituto de Geofísica, Departamento de Recursos Naturales
Universidad Nacional Autónoma de México, Mexico
alduncin@geofisica.unam.mx

Part I: Physical Systems

1 Concentration - Diffusion Dispersion Flux - Source Control
Constrained Transport Mixed Model

2 Mass Flux Rate - Pressure
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3 Velocity - Stress
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6 Dual Evolution Mixed Variational Flow

7 Dual Evolution Mixed Variational Deformation

1 Concentration - Diffusion Dispersion Flux - Source Control Mixed Constrained Transport Model

- *Balance of mass principle*

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{P}_t}\phi c\; d\Omega \;&=\; \int_{\mathcal{P}_t}\frac{\partial(\phi c)}{\partial t} + div(\phi \boldsymbol{u}_a\; c)\; d\Omega \\ &=\; -\int_{\mathcal{P}_t} s^*\; d\Omega + \int_{\mathcal{P}_t}\widehat{f}^*\; d\Omega - \int_{\partial\mathcal{P}_t}\boldsymbol{d}^{*}\cdot\boldsymbol{n}\; d\partial\Omega\end{aligned} \tag{1}$$

- *Primal evolution mixed control transport problem*

Find a primal-dual field $(c, \boldsymbol{d}^*, s^*)$ of concentration - flux - control

$$\left.\begin{aligned}&\frac{\partial(\phi c)}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{grad}\; c + div\boldsymbol{u}\; c + div\boldsymbol{d}^* = -s^* + \widehat{f}^* \\ &-\boldsymbol{D}(\boldsymbol{u})\boldsymbol{grad}\; c = \boldsymbol{d}^* \\ &c \in \partial\varphi^*(s^*)\end{aligned}\right\}\; \text{in } \Omega\times(0,T)$$

$$\begin{aligned}&c = \hat{c} && \text{on } \partial\Omega_{\tau_D}\times(0,T) \\ &\boldsymbol{d}^*\cdot\boldsymbol{n} = \hat{d}^*_n && \text{on } \partial\Omega_{\tau_N}\times(0,T) \\ &c(0) = \hat{c}_0 && \text{in } \Omega\end{aligned} \tag{2}$$

2 Mass Flux Rate - Pressure Mixed Compressible Darcian Flow Model

- *Balance of mass principle*

$$\frac{d}{dt}\int_{\mathcal{P}_t}\phi\rho\, d\Omega = \int_{\mathcal{P}_t}\frac{\partial(\phi\rho)}{\partial t} + div(\rho\,\phi\boldsymbol{u}_a)\, d\Omega = \int_{\mathcal{P}_t}\rho\hat{q}\, d\Omega - \int_{\partial\mathcal{P}_t}\alpha\boldsymbol{v}\cdot\boldsymbol{n}\, d\partial\Omega \quad (3)$$

- *Dual evolution mixed flow problem*

Find a primal-dual field $(\boldsymbol{w} = \rho\boldsymbol{u}, p^*)$ of mass flux - pressure

$$\left.\begin{aligned} &\mu(\rho\boldsymbol{K})^{-1}\boldsymbol{w} = -\boldsymbol{grad}\; p^* + \rho\boldsymbol{g} \\ &\phi\rho\vartheta\frac{\partial p^*}{\partial t} + div\;\boldsymbol{w} + \alpha\; div\boldsymbol{v} = \rho\hat{q} \end{aligned}\right\} \text{ in } \Omega\times(0,T)$$

$$\boldsymbol{w}\cdot\boldsymbol{n} = \widehat{w}_n \qquad \text{on } \partial\Omega_{f_N} \quad (4)$$

$$p^* = \hat{p}^* \qquad \text{on } \partial\Omega_{f_D}$$

$$p^*(0) = \widehat{p_0^*} \qquad \text{in } \Omega$$

3 Velocity - Stress
Mixed Elastoviscoplastic Deformation Model

- *Dual evolution quasistatic mixed deformation problem*

Find a primal-dual field $(\boldsymbol{v}, \boldsymbol{S}^*)$ of velocity-stress

$$\left.\begin{aligned} &\boldsymbol{div}\boldsymbol{S}^*(\boldsymbol{x},t) - \alpha\boldsymbol{grad}p^* + \boldsymbol{b}^*(\boldsymbol{x},t) \in \boldsymbol{\partial 0}(\boldsymbol{v}(\boldsymbol{x},t)) = \{\boldsymbol{0}\} \\ &\boldsymbol{\nabla}_s\boldsymbol{v}(\boldsymbol{x},t) - \boldsymbol{C}^{-1}(\boldsymbol{x})\frac{\partial \boldsymbol{S}^*}{\partial t}(\boldsymbol{x},t) \in \boldsymbol{\partial}\boldsymbol{\varphi}^*(\boldsymbol{x}\boldsymbol{S}^*(\boldsymbol{x},t)) \end{aligned}\right\} \quad (\boldsymbol{x},t) \in \Omega \times (0,T)$$

$$-\boldsymbol{S}^*(\boldsymbol{x},t)\boldsymbol{\nu}(\boldsymbol{x}) \in \boldsymbol{\partial}\psi_{DN}(\boldsymbol{v}(\boldsymbol{x},t)) = \boldsymbol{\partial I}_{\{\widehat{v}(x,t)\}}(\boldsymbol{v}(\boldsymbol{x},t)) \qquad (\boldsymbol{x},t) \in \partial\Omega_D \times (0,T)$$

$$-\boldsymbol{S}^*(\boldsymbol{x},t)\boldsymbol{\nu}(\boldsymbol{x}) \in \boldsymbol{\partial}\psi_{DN}(\boldsymbol{v}(\boldsymbol{x},t)) = \{\widehat{\boldsymbol{s}^*}(\boldsymbol{x},t)\} \qquad (\boldsymbol{x},t) \in \partial\Omega_N \times (0,T)$$

$$\boldsymbol{S}^*(\boldsymbol{x},0) = \boldsymbol{S}^*_{\boldsymbol{0}}(\boldsymbol{x}) \qquad \boldsymbol{x} \in \Omega \tag{5}$$

4 Variational Functional Frameworks

- *Stationary τ-transport, f-flow, d-deformation framework*

$\Omega \subset \Re^d$	spatial fixed bounded domain $d \in \{1,2,3\}$
$V_s(\Omega)$ and $Y_s^*(\Omega)$, $s \in \{\tau, f, d\}$	primal and dual Ω-field reflexive Banach spaces
$V_s^*(\Omega)$ and $Y_s(\Omega)$	topological duals
$H_s(\Omega)$ and $Z_s^*(\Omega)$	primal and dual Hilbert pivot spaces
$B_s(\partial\Omega)$ and $B_s^*(\partial\Omega)$	primal and dual reflexive Banach boundary space

- *Evolution reflexive Banach framework*

$\Omega \times (0,T)$, $T > 0$	space-time domain
$\mathcal{V}_s = L^p(0,T;V_s(\Omega))$, $2 \le p < \infty$	primal evolution space
$\mathcal{Y}_s^* = L^{q^*}(0,T;Y_s^*(\Omega))$, $q^* = p/(p-1)$	dual evolution space
$\mathcal{V}_s^* = L^{q^*}(0,T;V_s^*(\Omega))$, $\mathcal{Y}_s = L^p(0,T;Y_s(\Omega))$	topological duals
$\mathcal{B}_{s\partial\Omega} = L^p(0,T;B_s(\partial\Omega))$, $\mathcal{B}_{s\partial\Omega}^* = L^{q^*}(0,T;B_s^*(\partial\Omega))$	boundary evolution spaces

5 Primal Evolution Mixed Variational Transport

• *Concentration and normal flux maximal monotone subdifferential boundary variational equations* (Alduncin 1989)

$$\begin{aligned} \delta^*_{\tau_D} \boldsymbol{d}^* &\in \partial\psi_{\tau_D}(\pi_{\tau_D} c) = \partial I_{\{\hat{c}\}}(\pi_{\tau_D} c) \quad \text{in} \;\; B^*(\partial\Omega_{\tau_D}) \\ \delta^*_{\tau_N} \boldsymbol{d}^* &\in \partial\psi_{\tau_N}(\pi_{\tau_N} c) = \{\hat{d}^*_n\} \qquad \text{in} \;\; B^*(\partial\Omega_{\tau_N}) \end{aligned} \tag{6}$$

• *Primal trace compatibility property* (Girault-Raviart 1986)

$$(\boldsymbol{C}_{\boldsymbol{\pi_\tau}}) \; \pi_\tau \in \mathcal{L}(V_\tau(\Omega), B_\tau(\partial\Omega)) \text{ is surjective}$$

• *Compositional duality boundary variational equivalence* (Alduncin 2005)

$$\pi^T_{\tau_D} \partial I_{\{\hat{c}\}}(\pi_{\tau_D} c) \iff \partial(I_{\{\hat{c}\}} \circ \pi_{\tau_D})(c) \tag{7}$$

• *Primal variational divergence theorem*: for $\boldsymbol{y}^* \in \boldsymbol{Y}^*_{\boldsymbol{\tau}}(\Omega)$

$$div_\tau \, \boldsymbol{y}^* + grad^T_\tau \, \boldsymbol{y}^* = \pi^T_\tau \delta^*_\tau \, \boldsymbol{y}^* \quad \text{in } V^*_\tau(\Omega) \tag{8}$$

$$
(\boldsymbol{\mathcal{M}}_\tau)\begin{cases}
\text{Given } \widehat{f}_\tau^* \in \mathcal{V}_\tau^*, \hat{c}_0 \in H_\tau(\Omega), \text{ find } c \in \mathcal{W}_\tau, \boldsymbol{d}^* \in \boldsymbol{\mathcal{Y}}_\tau^* \text{ and } s^* \in \mathcal{V}_\tau^* : \\
grad_\tau^T \boldsymbol{d}^* - I_{\mathcal{V}_\tau}^T s^* \in \dfrac{d(\phi c)}{dt} + (\boldsymbol{u}\cdot\boldsymbol{grad}_\tau + div_\tau \boldsymbol{u})(c) \\
\quad +\partial(I_{\{\hat{c}\}} \circ \pi_{\tau_D})(c) - \widetilde{f}_\tau^* & \text{in } \mathcal{V}_\tau^* \\
-\boldsymbol{grad}_\tau\ c = \boldsymbol{D}_\tau^{-1}(\boldsymbol{d}^*) & \text{in } \boldsymbol{\mathcal{Y}}_\tau, \\
I_{\mathcal{V}_\tau} c \in \partial\varphi_\tau^*(s^*) & \text{in } \mathcal{V}_\tau, \\
c(0) = \hat{c}_0
\end{cases}
$$

• *Primal duality principle (Alduncin 2007): Under classical compatibility condition*

$$
(\boldsymbol{C}_{\varphi_\tau})\ \text{int}\ \mathcal{D}(\varphi_\tau) \neq \emptyset \implies \partial(\varphi_\tau \circ I_{\mathcal{V}_\tau}) = I_{\mathcal{V}_\tau}^T \partial\varphi_\tau \circ I_{\mathcal{V}_\tau}
$$

problem $(\boldsymbol{\mathcal{M}}_\tau)$ *is solvable if, and only if, primal problem*

$$
(\boldsymbol{\mathcal{P}}_\tau)\begin{cases}
\text{Find } c \in \mathcal{W}_\tau : \\
0 \in \dfrac{d(\phi c)}{dt} + (grad_\tau^T \boldsymbol{D}\ \boldsymbol{grad}_\tau + \boldsymbol{u}\cdot\boldsymbol{grad}_\tau)(c) + div_\tau \boldsymbol{u}(c) \\
\quad +\partial(I_{\{\hat{c}\}} \circ \pi_{\tau_D})(c) + \partial\varphi_\tau(c) - \widetilde{f}_\tau^*, \ \text{in } \mathcal{V}_\tau^* \\
c(0) = \hat{c}_0
\end{cases}
$$

is solvable

6 Dual Evolution Mixed Variational Flow

• *Normal mass flux rate and pressure maximal monotone subdifferential boundary variational equations*

$$\begin{aligned} \delta^*_{f_N} p^* &\in \partial\psi_{f_N}(\pi_{f_N}\boldsymbol{w}) = \partial I_{\{\hat{w}_n\}}(\pi_{f_N}\boldsymbol{w}) \;\; \text{in} \;\; B^*(\partial\Omega_{f_N}) \\ \delta^*_{f_D} p^* &\in \partial\psi_{f_D}(\pi_{f_D}\boldsymbol{w}) = \{\hat{p}^*\} \qquad\quad \text{in} \;\; B^*(\partial\Omega_{f_D}) \end{aligned} \tag{9}$$

• *Primal trace compatibility property*

$$(\boldsymbol{C}_{\boldsymbol{\delta_f}})\; \pi_f \in \boldsymbol{\mathcal{L}}(\boldsymbol{V_f}(\Omega), B_f(\partial\Omega)) \text{ is surjective}$$

• *Compositional duality boundary variational equivalence*

$$\boldsymbol{\pi}^T_{\boldsymbol{f_N}} \partial I_{\{\hat{w}_n\}}(\pi_{f_N}\boldsymbol{w}) \iff \boldsymbol{\partial}(I_{\{\hat{w}_n\}} \circ \pi_{f_N})(\boldsymbol{w}) \tag{10}$$

• *Primal variational divergence theorem*: for $\boldsymbol{y^*} \in \boldsymbol{Y^*_\tau}(\Omega)$

$$\boldsymbol{div^T_f} y^* + \boldsymbol{grad}_f y^* = \boldsymbol{\pi^T_f \delta^*_f} y^* \;\; \text{in} \boldsymbol{V^*_f}(\Omega) \tag{11}$$

$$
(\boldsymbol{\mathcal{M}}_f^*)\begin{cases}\text{Given } \hat{q} \in \mathcal{Y}_f, \widehat{p_0^*} \in L^2(\Omega), \text{ find } \boldsymbol{w} \in \boldsymbol{\mathcal{V}_f} \text{ and } p^* \in \mathcal{X}_f^* : \\ \boldsymbol{div}_f^T p^* \in \boldsymbol{\partial F_f}(\boldsymbol{w}) - \boldsymbol{f_f^*} & \text{in } \boldsymbol{\mathcal{V}_f^*} \\ -div_f \boldsymbol{w} = A_f(p^*)\dfrac{\partial p^*}{\partial t} + \alpha \, div\boldsymbol{v} - \rho\hat{q} & \text{in } \mathcal{Y}_f \\ p^*(0) = \widehat{p_0^*} \end{cases}
$$

• *Dual duality principle: Under compatibility condition*

$(\boldsymbol{C}_{\boldsymbol{F_f^*},\boldsymbol{div_f^T}})\ \mathrm{int}\mathcal{D}(F_f^*)\cap\mathcal{R}(\boldsymbol{div}_f^T) \neq \emptyset \implies \partial(F_f^* \circ \boldsymbol{div}_f^T) = \boldsymbol{div}_f \partial F_f^* \circ \boldsymbol{div}_f^T,$

problem $(\boldsymbol{\mathcal{M}}_f^*)$ *is solvable if, and only if, dual problem*

$$
(\boldsymbol{\mathcal{D}}_f^*)\begin{cases}\textit{Find } p^* \in \mathcal{X}_f^* : \\ 0 \in A_f(p^*)\dfrac{\partial p^*}{\partial t} + \boldsymbol{\partial}(F_f^* \circ \boldsymbol{div}^T)(p^* + r_{f*}^*) + \alpha \, div\boldsymbol{v} - \rho\hat{q} \textit{ in } \mathcal{Y}_f \\ p^*(0) = \widehat{p_0^*} \end{cases}
$$

is solvable

7 Dual Evolution Mixed Variational Deformation

• *Velocity and traction maximal monotone subdifferential boundary variational equations*

$$\begin{aligned} \boldsymbol{\delta^*}_{d_D} \boldsymbol{S}^* \in \partial\psi_{d_D}(\boldsymbol{\pi}_{d_D}\boldsymbol{v}) = \partial I_{\{\widehat{\boldsymbol{v}}\}}(\boldsymbol{\pi}_{d_D}\boldsymbol{v}) \quad \text{in} \quad \boldsymbol{B}^*(\partial\Omega_{d_D}) \\ \boldsymbol{\delta^*}_{d_N} \boldsymbol{S}^* \in \partial\psi_{d_N}(\boldsymbol{\pi}_{d_N}\boldsymbol{v}) = \{\widehat{\boldsymbol{s}^*}\} \qquad \text{in} \quad \boldsymbol{B}^*(\partial\Omega_{d_N}) \end{aligned} \tag{12}$$

• *Primal trace compatibility property*

$$(\boldsymbol{C1}_{\boldsymbol{\pi_d}})\ \boldsymbol{\pi_d} \in \boldsymbol{\mathcal{L}}(\boldsymbol{V_d}(\Omega), \boldsymbol{B_d}(\partial\Omega)) \text{ is surjective}$$

• *Compositional duality boundary variational equivalence*

$$\boldsymbol{\pi}^T_{\boldsymbol{d_D}} \boldsymbol{\partial I}_{\{\widehat{v}\}}(\boldsymbol{\pi_{d_D}}\boldsymbol{v}) \iff \boldsymbol{\partial}(I_{\{\widehat{v}\}} \circ \boldsymbol{\pi_{d_D}})(\boldsymbol{v}) \tag{13}$$

• *Primal variational divergence theorem*: for $\boldsymbol{T}^* \in \boldsymbol{Y}^*_{\boldsymbol{\tau}}(\Omega)$

$$\boldsymbol{H}^T_d \boldsymbol{T}^* + \boldsymbol{D}_d \boldsymbol{T}^* = -\boldsymbol{\pi}^T_{\boldsymbol{d}} \boldsymbol{\delta}^*_{\boldsymbol{d}} \boldsymbol{T}^* \ \text{ in } \boldsymbol{V}^*_{\boldsymbol{d}}(\Omega) \tag{14}$$

$$(\boldsymbol{\mathcal{M}}_d^*)\begin{cases}\text{Find } \boldsymbol{v} \in \boldsymbol{\mathcal{V}}_d \text{ and } \boldsymbol{S}^* \in \boldsymbol{\mathcal{X}}_{\boldsymbol{d}}^* : \\ -\boldsymbol{H}_d^T \boldsymbol{S}^* \in \boldsymbol{\partial F_d}(\boldsymbol{v}) - \widetilde{\boldsymbol{b}_{\boldsymbol{d}}^*}(p^*) \ \text{ in } \boldsymbol{\mathcal{V}}_d^* \\ \boldsymbol{H}_d \boldsymbol{v} \in \boldsymbol{A}_d \dfrac{d\boldsymbol{S}^*}{dt} + \boldsymbol{\partial \Phi_d^*}(\boldsymbol{S}^*) \ \text{ in } \boldsymbol{\mathcal{Y}}_d \\ \boldsymbol{S}^*(0) = \boldsymbol{S}_{\boldsymbol{0}}^* \end{cases}$$

• *Dual duality principle: Under compatibility condition*

$(\boldsymbol{C}_{\boldsymbol{F_d^*}, -\boldsymbol{H_d^T}})\ \text{int}\mathcal{D}(F_d^*) \cap \mathcal{R}(-\boldsymbol{H}_d^T) \neq \emptyset \implies \boldsymbol{\partial}(F_d^* \circ (-\boldsymbol{H}_d^T)) = -\boldsymbol{H}_d \partial F_d^* \circ (-\boldsymbol{H}_d^T)$

problem $(\boldsymbol{\mathcal{M}}_d^*)$ *is solvable if, and only if, dual problem*

$$(\boldsymbol{\mathcal{D}}_d)\begin{cases}\textit{Find } \boldsymbol{S}^* \in \boldsymbol{\mathcal{X}}_d^* : \\ \boldsymbol{0} \in \boldsymbol{A}_d \dfrac{d\boldsymbol{S}^*}{dt} + \boldsymbol{\partial \Phi_d^*}(\boldsymbol{S}^*) + \boldsymbol{\partial}(F_d^* \circ (-\boldsymbol{H}_d^T))(\boldsymbol{S}^* + \boldsymbol{R}_{\widetilde{\boldsymbol{b}_{\boldsymbol{d}}^*}}) \ \textit{ in } \boldsymbol{\mathcal{Y}}_d \\ \boldsymbol{S}^*(0) = \boldsymbol{S}_{\boldsymbol{0}}^* \end{cases}$$

is solvable

8 Norton-Hoff Model

• *Nonlinear Maxwell viscoelastic material with conjugate dissipation or yield convex superpotential*

$$\varphi^*(\boldsymbol{x}, \boldsymbol{S}^*) = \frac{\lambda}{q}\Big\| \inf_{\mu>0}\{\mu : \boldsymbol{S}^*_{\boldsymbol{dev}} \in \mu \boldsymbol{\mathcal{P}}(\boldsymbol{x})\Big\|^q \tag{15}$$

$\lambda \in L^\infty(\Omega), \;\; 1 < q < \infty$

$\boldsymbol{S}^*_{\boldsymbol{dev}} = \boldsymbol{S}^* - (1/n)\; tr(\boldsymbol{S}^*)\boldsymbol{I}$ deviatoric part of of the stress $\boldsymbol{S}^*$

$\boldsymbol{\mathcal{P}}$ convex set of admissible stress fields (gauge function)

• (LeTallec 1990) *Under condition*

$$(\boldsymbol{C}_{\varphi^*})\; C_2\|\boldsymbol{S}^*_{\boldsymbol{dev}}\|^q \leq \varphi^*(\boldsymbol{x}, \boldsymbol{S}^*) = \varphi^*(\boldsymbol{x}, \boldsymbol{S}^*_{\boldsymbol{dev}}) \leq C_1\|\boldsymbol{S}^*_{\boldsymbol{dev}}\|^q$$

dual evolution mixed variational deformation problem $(\boldsymbol{\mathcal{M}}^*_d)$ *is solvable relative to the mixed velocity-stress spatial functional framework, for* $2 \leq p < +\infty$ *and* $q^* = p/(p-1)$

$$\begin{aligned} \boldsymbol{V}(\Omega) &= \{\boldsymbol{v} \in \boldsymbol{W}^{1,q^*}(\Omega) : \; div\; \boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\} \\ \boldsymbol{Y}^*(\Omega) &= \{\boldsymbol{T} : tr(\boldsymbol{T}) \in \boldsymbol{L}^2(\Omega), \boldsymbol{T}_d \in \boldsymbol{L}^p(\Omega)\} \end{aligned} \tag{16}$$

9 Perzyna Model

• *Elastic-viscoplastic constitutive model with differentiable conjugate dissipation or yield functional (Perzyna 1966)*

$$\varphi^*(\boldsymbol{S}^*) = \frac{1}{2\nu}\|\boldsymbol{S}^* - \boldsymbol{\Pi}_{\mathcal{P}}(\boldsymbol{S}^*)\|^2 \tag{17}$$

$\nu > 0$ viscosity parameter

$\boldsymbol{\Pi}_{\mathcal{P}}$ orthogonal projection into admissible Cauchy stress convex set $\boldsymbol{\mathcal{P}}$

• (Rochdi-Sofonea 1997) *On usual mixed velocity and stress field Hilbert spaces*

$$\begin{gathered} \boldsymbol{V}(\Omega) = \boldsymbol{H}^1(\Omega) \\ \boldsymbol{Y}(\Omega) = \boldsymbol{H}(\boldsymbol{div};\Omega) = \{\boldsymbol{T} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{T} = \boldsymbol{T}^T, \boldsymbol{div}\,\boldsymbol{T} \in \boldsymbol{L}^2(\Omega)\} \end{gathered} \tag{18}$$

dual evolution mixed deformation problem $\boldsymbol{\mathcal{M}}_d^*$ *is solvable*

10 Elastic-Perfectly-Plastic-Model

• *Elastic perfectly plastic material with conjugate dissipation or yield convex superpotential*

$$\varphi^*(\boldsymbol{S}^*) = I_{\mathcal{P}}(\boldsymbol{S}^*) \tag{19}$$

$\boldsymbol{\mathcal{P}}$ closed convex set of elastic stress fields

$\partial\boldsymbol{\mathcal{P}}$ the yield limit

• (Sofonea-Renon-Shilor 2004) *In the mixed velocity-stress Hilbert framework*

$$\begin{aligned} \boldsymbol{V}(\Omega) &= \boldsymbol{H}^1(\Omega) \\ \boldsymbol{Y}(\Omega) &= \{\boldsymbol{T} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{T} = \boldsymbol{T}^T\} \end{aligned} \tag{20}$$

the solvability of evolution dual problem $\boldsymbol{\mathcal{D}}_d$, *and consequently, the solvability of corresponding mixed problem* $(\boldsymbol{\mathcal{M}}_d^*)$ *is guaranteed*

11 Variational Macro-Hybrid Functional Framework

• *Nonoverlapping domain decomposition*

$$\overline{\Omega} = \bigcup_{e=1}^{E} \overline{\Omega}_e \qquad \Omega_e \cap \Omega_f = \emptyset \quad 1 \leq e < f \leq E$$

$$\Gamma_e = \partial\Omega_e \cap \Omega \quad 1 \leq e \leq E \;\; \text{(internal boundaries)} \tag{21}$$

$$\Gamma_{ef} = \Gamma_e \cap \Gamma_f \;\; 1 \leq e < f \leq E \;\; \text{(interfaces)}$$

• *Decomposable reflexive Banach stationary mixed framework*

$$\boldsymbol{V}_s(\Omega) \simeq \left\{ \boldsymbol{v} \in \boldsymbol{V}_s(\{\Omega_e\}) \equiv \prod_{e=1}^{E} \boldsymbol{V}_s(\Omega_e) : \{\pi_{s_e}\boldsymbol{v}\} \in \boldsymbol{Q}_s \subset \boldsymbol{B}_s(\{\Gamma_{s_e}\}) \right\}$$

$$\boldsymbol{Y}_s^*(\Omega) \simeq \boldsymbol{Y}_s^*(\{\Omega_e\}) \equiv \prod_{e=1}^{E} \boldsymbol{Y}_s^*(\Omega_e) \tag{22}$$

• *Dual internal boundary transmission polar subspace*

$$\boldsymbol{Q}_s^* = \left\{ \{\mu_e^*\} \in \boldsymbol{B}_s^*(\{\Gamma_{s_e}\}) : \; \langle \{\mu_e^*\}, \{\mu_e\} \rangle_{B_s(\{\Gamma_{s_e}\})} = 0, \forall \{\mu_e\} \in \boldsymbol{Q}_s \right\} \tag{23}$$

• *Under the fundamental macro-hybrid compatibility condition* (Alduncin 2007)

$$(\boldsymbol{C}_{[\pi_{\Gamma_e}]}) \;\; [\boldsymbol{\pi}_{\Gamma_e}] \in (\boldsymbol{\mathcal{L}}(\boldsymbol{V}(\{\Omega_e\}), \boldsymbol{B}(\{\Gamma_e\})) \;\; \text{is surjective}$$

$$\{\lambda_{s_e}^*\} \in \boldsymbol{\partial} I_{Q_s} \circ [\pi_{s_{\Gamma_e}}](\{u_{s_e}\}) \Longleftrightarrow \{\pi_{s_{\Gamma_e}} u_{s_e}\} \in \boldsymbol{\partial} \boldsymbol{I}_{Q^*}(\{\lambda_{s_e}^*\}) \tag{24}$$

12 Macro-Hybrid Primal Mixed Variational Transport

$$
(\boldsymbol{\mathcal{M}}_{\tau_e})\begin{cases}
\text{For } 1 \le e \le E, \text{ find } c_e \in \mathcal{W}_{\tau_{\Omega_e}}, \boldsymbol{d_e^*} \in \boldsymbol{\mathcal{Y}^*}_{\boldsymbol{\tau_{\Omega_e}}} \text{ and } s_e^* \in \mathcal{V}^*{}_{\tau_{\Omega_e}} : \\
grad_{\tau_e}^T \boldsymbol{d_e^*} - I_{\mathcal{V}_{\tau_{\Omega_e}}}^T s_e^* - \pi_{\tau_{\Gamma_e}}^T \lambda_{\tau_e}^* \in \dfrac{d(\phi_e c_e)}{dt} + (\boldsymbol{u_e} \cdot \boldsymbol{grad_{\tau_e}} + div_{\tau_e} \boldsymbol{u_e})(c_e) \\
\quad + \partial(I_{\{\hat{c}_e\}} \circ \pi_{\tau_{eD}})(c_e) - \widetilde{f}_{\tau_e}^* \qquad\qquad \text{in } \mathcal{V}^*{}_{\tau_{\Omega_e}} \\
\quad -\boldsymbol{grad_{\tau_e}}\, c_e = \boldsymbol{D_{\tau_e}}^{-1}(\boldsymbol{d_e^*}) \qquad\qquad \text{in } \boldsymbol{\mathcal{Y}}_{\boldsymbol{\tau_{\Omega_e}}} \\
\quad I_{\mathcal{V}_{\tau_{\Omega_e}}} c_e \in \partial \varphi_{\tau_e}^*(s_e^*) \qquad\qquad \text{in } \mathcal{V}_{\tau_{\Omega_e}} \\
c_e(0) = \hat{c}_{e_0}
\end{cases}
$$

synchronized by dual transmission problem

$$
(\boldsymbol{\mathcal{T}}_\tau^*)\begin{cases}
\text{Find } \{\lambda_{\tau_e}^*\} \in \boldsymbol{\mathcal{B}}^*_{\tau_{\{\boldsymbol{\Gamma_e}\}}} : \\
\{\pi_{\tau_{\Gamma_e}} c_e\} \in \boldsymbol{\partial I}_{\boldsymbol{Q_\tau^*}}(\{\lambda_{\tau_e}^*\}) \qquad\qquad \text{in } \boldsymbol{\mathcal{B}}_{\tau_{\{\boldsymbol{\Gamma_e}\}}}
\end{cases}
$$

13 Macro-Hybrid Dual Mixed Variational Flow

$$(\boldsymbol{\mathcal{M}}_f^*) \begin{cases} \text{For } 1 \leq e \leq E, \text{ find } \boldsymbol{w_e} \in \boldsymbol{\mathcal{V}}_{\boldsymbol{f}_{\boldsymbol{\Omega_e}}} \text{ and } p_e^* \in \mathcal{X}_{f_{\Omega_e}}^* : \\ \boldsymbol{div}_{f_e}^T p_e^* - \boldsymbol{\pi}_{\boldsymbol{e}_{\boldsymbol{\Gamma_e}}}^{\boldsymbol{T}} \lambda_{d_e}^* \in \boldsymbol{\partial F}_{\boldsymbol{f_e}}(\boldsymbol{w}_e) - \boldsymbol{f}_{\boldsymbol{f_e}}^* & \text{in } \boldsymbol{\mathcal{V}}_{\boldsymbol{f}_{\boldsymbol{\Omega_e}}}^* \\ -div_{f_e} \boldsymbol{w_e} = A_{f_e}(p_e^*) \dfrac{\partial p_e^*}{\partial t} + \alpha_e \, div_{f_e} \boldsymbol{v_e} - \rho_e \hat{q}_e & \text{in } \mathcal{Y}_{f_{\Omega_e}} \\ p_e^*(0) = \widehat{p_{0_e}^*} \end{cases}$$

synchronized by dual transmission problem

$$(\boldsymbol{\mathcal{T}}_f^*) \begin{cases} \text{Find } \{\lambda_{d_e}^*\} \in \boldsymbol{\mathcal{B}}_{\boldsymbol{f}_{\{\boldsymbol{\Gamma_e}\}}}^* : \\ \{\pi_{e_{\Gamma_e}} \boldsymbol{w}_e\} \in \boldsymbol{\partial I}_{Q_{\boldsymbol{f}}^*}(\{\lambda_{d_e}^*\}) \ \text{ in } \boldsymbol{\mathcal{B}}_{\boldsymbol{f}_{\{\boldsymbol{\Gamma_e}\}}} \end{cases}$$

14 Elastoviscoplastic Multi-Constitutive Models

• *Macro-hybrid localized elastoviscoplastic constitutivities models of dual evolution mixed deformation problem* $(\boldsymbol{\mathcal{M}}_d^*)$

$$(\boldsymbol{\mathcal{M}}_{\boldsymbol{d_e}}^*)\begin{cases}\text{For } 1 \le e \le E, \text{ find } (\boldsymbol{v}_e, \boldsymbol{S}_e^*) \in \boldsymbol{\mathcal{V}}_{\mathbf{d}_{\boldsymbol{\Omega_e}}} \times \boldsymbol{\mathcal{X}}^*_{\mathbf{d}_{\boldsymbol{\Omega_e}}} : \\ -\boldsymbol{H}_{d_e}^T \boldsymbol{S}_{\boldsymbol{e}}^* - \boldsymbol{\pi}_{\mathbf{d}_{\boldsymbol{\Gamma_e}}}^T \boldsymbol{\lambda}_{\boldsymbol{d_e}}^* \in \boldsymbol{\partial F}_{\boldsymbol{d_e}}(\boldsymbol{v}_e) - \widetilde{\boldsymbol{b}_{\boldsymbol{d_e}}^*}(p_e^*) & \text{in } \boldsymbol{\mathcal{V}}^*_{\mathbf{d}_{\boldsymbol{\Omega_e}}} \\ \boldsymbol{H}_{d_e} \boldsymbol{v}_e \in \boldsymbol{A}_{d_e} \dfrac{d\boldsymbol{S}_{\boldsymbol{e}}^*}{dt} + \boldsymbol{\partial \Phi}_{\boldsymbol{d_e}}^*(\boldsymbol{S}_e^*) & \text{in } \boldsymbol{\mathcal{Y}}_{\mathbf{d}_{\boldsymbol{\Omega_e}}} \\ \boldsymbol{S}_e^*(0) = \boldsymbol{S}_{\mathbf{0_e}}^* \end{cases}$$

synchronized by dual transmission problems

$$(\boldsymbol{\mathcal{T}}_d^*)\begin{cases}\text{Find } \{\boldsymbol{\lambda}_{\boldsymbol{d_e}}^*\} \in \boldsymbol{\mathcal{B}}^*_{\boldsymbol{d}_{\{\boldsymbol{\Gamma_e}\}}} : \\ \{\boldsymbol{\pi}_{\mathbf{d}_{\boldsymbol{\Gamma_e}}} \boldsymbol{v}_e\} \in \boldsymbol{\partial I}_{\boldsymbol{Q_d^*}}(\{\boldsymbol{\lambda}_{\boldsymbol{d_e}}^*\}) \ \text{ in } \boldsymbol{\mathcal{B}}_{\boldsymbol{d}_{\{\boldsymbol{\Gamma_e}\}}}\end{cases}$$

Conclusions

- *Variational Modeling and Existence Analysis of Transport-Flow through Elastoviscoplastic Porous Media*
- *Classical and Surjectivity Coupling Compatibility Conditions for Compositional Dualization*
- *Primal and Dual Duality Principles*
- *Variational Macro-Hybridization for Localization, Scaling, Multi-Constitutivity, Multi-Algorithmia and Parallel Computing*
- *Proximation Realization of Semi-Implicit Time Marching Schemes*
- *Mixed and Macro–Hybrid Variational Basis for Internal Variational Fully Discrete Approximations*